Int. J. Solids Structures Vol. 23, No. 3, pp. 335–349, 1987
Printed in Great Britain.

0020-7683/87 $3.00+.00
Pergamon Journals Ltd.

ON THE NUMERICAL SOLUTION OF NON-LINEAR STRING PROBLEMS USING THE THEORY OF A COSSERAT POINT

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(*Received 22 May* 1985; *in revised form* 1 *November* 1985)

Abstract—The objective of this paper is to develop an appropriate form of the theory of a Cosserat point (*ASME J. Appl. Mech.* **52**, 368 (1985)), which can be used to formulate the numerical solution of the three-dimensional motion of a non-linear elastic string. The string is divided into N material parts, each of which is modelled as a Cosserat point with its own equations of motion and constitutive equations. Then the motion of each Cosserat point is coupled with that of its neighbours and boundary conditions are introduced to obtain a system of ordinary differential equations of time only which describe the motion of the string. Two examples of a rotating string are considered. For each example we show that director inertia (rotary inertia) is significant and that the Cosserat solution converges rapidly to the exact solution developed by Rosenau and Rubin (*Phys. Rev.* **A31**, 3480 (1985)).

INTRODUCTION

Recently[1], the theory of Cosserat point was developed to analyse a body which is essentially a material point surrounded by some finite volume. This theory has been successfully used to formulate the numerical solution of one-dimensional continuum problems[2] and to analyse free vibrations of a rectangular parallelepiped[3]. In Ref. [2] the continuum was restricted to move in a single fixed direction and the non-linear motion of a bar was analysed.

The objective of this paper is to develop a theory of a Cosserat point which is applicable to the numerical solution of the non-linear three-dimensional motion of a string. The string is divided into N material parts, each of which is modelled as a Cosserat point with its own kinematics, kinetics and constitutive equations. A system of ordinary differential equations of time only, which describe the motion of the string, are obtained by coupling the motion of each Cosserat point to that of its neighbours and by imposing appropriate boundary conditions at the ends of the string. Actually, the string is modelled as a chain of vectors $\mathbf{d}_I$ $(I = 1, 2, \ldots, N)$ called directors (Fig. 1), each of which is allowed to move three-dimensionally in space.

In the following sections we discuss the basic equations of the theory of a Cosserat point developed by a direct approach. For clarity, we develop the same equations in Appendix A starting with the basic description of a string as a Cosserat curve. The utility of the theory is demonstrated by considering two examples of a non-linear elastic string. The first example analyses a rotating closed circular string and the second example analyses a rotating straight string. For each example the Cosserat solution is compared with the exact solution which was developed by Rosenau and Rubin[4] and which is recorded in Appendices B and C.

It is natural to ask what advantages a numerical solution procedure based on the theory of a Cosserat point has over other discretization techniques. In this regard, we emphasize that the theory of a Cosserat point is a continuum model which parallels the three-dimensional theory in that it is inherently nonlinear, valid for arbitrary materials, and invariant under superposed rigid body motions. Other procedures used to discretize a set of partial differential equations may or may not be endowed with these features at the element level. Obviously, for certain simple problems such as that considered by Rubin[2], the Cosserat procedures will produce similar results to those of other procedures. However, it is expected that once Cosserat points are developed for two- and three-

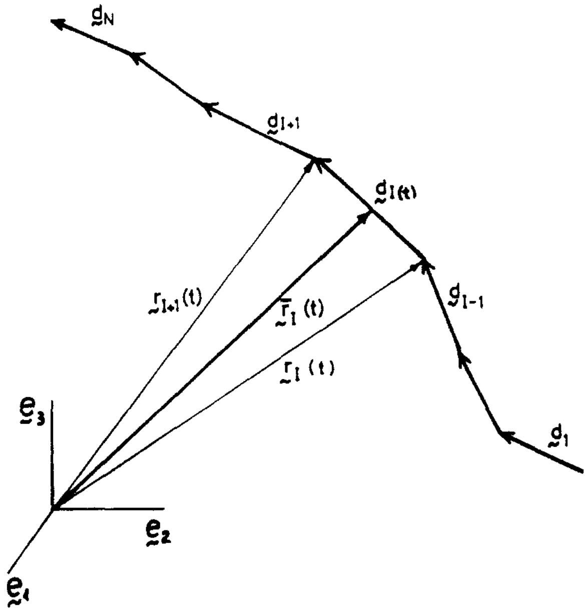

Fig. 1. Sketch of a string modelled by N Cosserat points. The Ith Cosserat point is characterized by its: position $\bar{\mathbf{r}}_I$, director $\mathbf{d}_I$, and end points $\mathbf{r}_I$ and $\mathbf{r}_{I+1}$.

dimensional problems they will provide guidance for developing potentially better finite elements or at least for examining the advantages of one finite element over another. Unfortunately, the analysis of these issues must await further theoretical developments.

BASIC EQUATIONS (DIRECT APPROACH)

Here, we are concerned with developing a theory of a Cosserat point which can be used to formulate the numerical solution of non-linear string problems. Basically, we divide the string into N material parts and model each part as a Cosserat point. In the present configuration, at time t, the Ith Cosserat point is characterized by its position $\bar{\mathbf{r}}_I(t)$, relative to the origin of a fixed coordinate system, and by a director $\mathbf{d}_I(t)$ which determines the length and orientation of the Cosserat point (Fig. 1). The position vector $\bar{\mathbf{r}}_I$ and director $\mathbf{d}_I$ are each three-dimensional vector functions of time only which in the reference configuration acquire the values $\bar{\mathbf{r}}_I = \bar{\mathbf{R}}_I$ and $\mathbf{d}_I = \mathbf{D}_I$. A motion of the Cosserat point is defined by†

$$\bar{\mathbf{r}}_I = \bar{\mathbf{r}}_I(t), \qquad \mathbf{d}_I = \mathbf{d}_I(t) \tag{1a, b}$$

$$d_I = (\mathbf{d}_I \cdot \mathbf{d}_I)^{1/2} > 0 \tag{1c}$$

where condition (1c) ensures that the length of the Cosserat point never vanishes. The velocity $\mathbf{v}_I$ and director velocity $\mathbf{w}_I$ are then calculated by

$$\bar{\mathbf{v}}_I = \dot{\bar{\mathbf{r}}}_I, \qquad \mathbf{w}_I = \dot{\mathbf{d}}_I \tag{2a, b}$$

where a superposed dot denotes time differentiation. Furthermore, the ends of the Ith Cosserat point are located by the position vectors $\mathbf{r}_I(t)$ and $\mathbf{r}_{I+1}(t)$ such that (Fig. 1)

$$\mathbf{d}_I = \mathbf{r}_{I+1} - \mathbf{r}_I. \tag{3}$$

† There is no summation over repeated capital indices which are used to indicate, say, the Ith Cosserat point.

We now turn to a statement of the conservation and balance laws of the Ith Cosserat point and, with reference to the present configuration, define the following quantities: the mass $m_I(t)$ of the point; the contact force $\mathbf{n}_{I1}(t)$ and contact moment† $\mathbf{m}_{I1}(t)$ applied to the end $\mathbf{r}_I$; the contact force $\mathbf{n}_{I2}(t)$ and contact moment $\mathbf{m}_{I2}(t)$ applied to the end $\mathbf{r}_{I+1}$; the assigned external force $\mathbf{f}_I(t)$ and assigned external director couple $\mathbf{l}_I(t)$; the intrinsic director couple $\mathbf{k}_I(t)$ which makes no contribution to the supply of angular momentum; and the constant inertia coefficients y_I^{11}, y_I^{11}. With the above definitions, we postulate the following conservation and balance laws for the Cosserat point under consideration:

$$\frac{\mathrm{d}}{\mathrm{d}t}[m_I] = 0 \tag{4a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[m_I(\bar{\mathbf{v}}_I + y_I^1\mathbf{w}_I)] = \mathbf{f}_I + \mathbf{n}_{I1} + \mathbf{n}_{I2} \tag{4b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[m_I(y_I^1\bar{\mathbf{v}}_I + y_I^{11}\mathbf{w}_I)] = \mathbf{l}_I + \mathbf{m}_{I1} + \mathbf{m}_{I2} - \mathbf{k}_I \tag{4c}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[\bar{\mathbf{r}}_I \times m_I(\bar{\mathbf{v}}_I + y_I^1\mathbf{w}_I) + \mathbf{d}_I \times (y_I^1\bar{\mathbf{v}}_I + y_I^{11}\mathbf{w}_I)]$$
$$= \bar{\mathbf{r}}_I \times [\mathbf{f}_I + \mathbf{n}_{I1} + \mathbf{n}_{I2}] + \mathbf{d}_I \times [\mathbf{l}_I + \mathbf{m}_{I1} + \mathbf{m}_{I2}]. \tag{5}$$

Equation (4a) represents the conservation of mass, eqn (4b) the balance of linear momentum, eqn (4c) the balance of director momentum, and eqn (5) the balance of angular momentum. Now, using eqns (4a)–(4c), the balance of angular momentum (5) may be written in the form

$$\mathbf{d}_I \times \mathbf{k}_I = 0 \tag{6}$$

which states that the intrinsic director couple $\mathbf{k}_I$ is parallel to $\mathbf{d}_I$ so that

$$\mathbf{k}_I = k_I\left(\frac{\mathbf{d}_I}{d_I}\right) \tag{7}$$

where the function k_I must be specified by a constitutive equation.

The equations of motion, eqns (4) and (5), may be obtained directly from the development in Ref. [1] by suppressing the effect of the directors $\mathbf{d}_2$ and $\mathbf{d}_3$ as well as their associated kinetic quantities. They also can be obtained by starting with a three-dimensional description of the string and introducing appropriate kinematic assumptions. Still another approach is to start with a model of a string as a Cosserat curve and introduce appropriate kinematic assumptions. This latter approach is described in detail in Appendix A. Here, we merely recall that material points of the string are identified by a convected coordinate ξ and are located by the position vector $\mathbf{r}^*(\xi, t)$. The material part associated with the Ith Cosserat point occupies the region defined by $\xi \in [\xi_I, \xi_{I+1}]$. Now, to establish a one-to-one correspondence with this theory of a Cosserat point and the description of the string as a Cosserat curve we introduce the kinematic assumption (A4). It follows that the ends $\mathbf{r}_I$ and $\mathbf{r}_{I+1}$ of the Ith Cosserat point are given by

$$\mathbf{r}_I = \bar{\mathbf{r}}_I - \frac{1}{2}\mathbf{d}_I, \qquad \mathbf{r}_{I+1} = \bar{\mathbf{r}}_I + \frac{1}{2}\mathbf{d}_I \tag{8a, b}$$

†The use of the words moment and couple arise because $\mathbf{m}_{I1}$, $\mathbf{m}_{I2}$ and $\mathbf{l}_I$ are related to forces times the coordinate θ_I (see eqns (A6)). Here θ_I is unitless so these quantities have the units of force.

so that

$$\bar{\mathbf{r}}_I = \frac{1}{2}(\mathbf{r}_I + \mathbf{r}_{I+1}). \tag{9}$$

One of the main advantages of using the direct approach over approximations from either the three-dimensional theory or the theory of a Cosserat curve is realized in the discussion of constitutive equations. For the direct approach we follow the procedures used in the three-dimensional theory and define the mechanical power P_I by

$$\begin{aligned} P_I = {} & (\mathbf{f}_I \cdot \bar{\mathbf{v}}_I + \mathbf{l}_I \cdot \mathbf{w}_I) + (\mathbf{n}_{I1} \cdot \bar{\mathbf{v}}_I + \mathbf{m}_{I1} \cdot \mathbf{w}_I) \\ & + (\mathbf{n}_{I2} \cdot \bar{\mathbf{v}}_I + \mathbf{m}_{I2} \cdot \mathbf{w}_I) \\ & - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2} m_I(\bar{\mathbf{v}}_I \cdot \bar{\mathbf{v}}_I + 2y_I^1 \bar{\mathbf{v}}_I \cdot \mathbf{w}_I + y_I^{11} \mathbf{w}_I \cdot \mathbf{w}_I)\right]. \end{aligned} \tag{10}$$

Using eqns (1c), (4) and (7) expression (10) for the mechanical power P_I can be rewritten in the reduced form

$$P_I = \mathbf{k}_I \cdot \mathbf{w}_I = k_I \dot{d}_I. \tag{11}$$

For a non-linear elastic Cosserat point we assume the existence of a strain energy function ψ_I such that

$$P_I = \dot{\psi}_I, \qquad \psi_I = \hat{\psi}_I(d_I, D_I) \tag{12a, b}$$

where D_I is the reference value of d_I. With the help of eqns (11) and (12), and assuming that for an elastic material k_I depends only on d_I and D_I, we deduce that

$$k_I = \frac{\partial \hat{\psi}_I}{\partial d_I}. \tag{13}$$

Within the context of this description of a Cosserat point we observe that the assigned force $\mathbf{f}_I$ and director couple $\mathbf{l}_I$ include contributions from external body forces applied to material points of the string as well as from surface tractions applied to the lateral surface of the string. Each of the quantities introduced for the Cosserat point is related directly to the description of the Cosserat curve (see eqns (A6)). In particular we recall (eqns (A6f) and (A6g)) that for this simple theory the contact moments $\mathbf{m}_{I1}$ and $\mathbf{m}_{I2}$ are determined by

$$\mathbf{m}_{I1} = -\frac{1}{2}\mathbf{n}_{I1}, \qquad \mathbf{m}_{I2} = \frac{1}{2}\mathbf{n}_{I2}. \tag{14a, b}$$

Furthermore, if in the reference configuration the mass density ρ_0^* (mass per unit length) of the string is constant and A^* in eqn (A3a) equals unity then from eqns (A3a), (A4b) and (A6a)–(A6c) we obtain the results

$$m_I = \rho_0^*(\xi_{I+1} - \xi_I), \qquad y_I^1 = 0, \qquad y_I^{11} = \frac{1}{12} \tag{15a–c}$$

where ξ_I now has the units of length. The inertia coefficient y_I^{11} is associated with director inertia which is more commonly called rotary inertia in rod and shell theory. Since we wish to examine the influence of director inertia on the solution of the examples considered later it is more convenient to write

$$y_I^{11} = \frac{\alpha}{12} \qquad (\alpha = 0, 1) \tag{16}$$

where for $\alpha = 1$ director inertia is included and for $\alpha = 0$ it is excluded.

SOLUTION PROCEDURE

An approximate solution for the three-dimensional motion of a string can be obtained by modelling the string as a chain of N Cosserat points (Fig. 1), with end points located by r_I $(I = 1, 2, \ldots, N + 1)$. Once a constitutive equation is specified for k_I in eqn (13) eqns (4) characterize the motion of the Ith Cosserat point. Following the procedure described in Ref. [2] the motion of the Ith Cosserat point may be coupled with the motion of its neighbours by introducing the $(N - 1)$ kinetic coupling equations

$$\mathbf{n}_{I-1,2} + \mathbf{n}_{I1} = 0 \qquad (I = 2, 3, \ldots, N). \tag{17}$$

Additional kinematic coupling is implied by relations (3) and (9).

For an arbitrary constitutive equation, eqn (13), and arbitrary values of the assigned force and assigned director couple the equations of motion, eqns (4), can be solved for the contact forces $\mathbf{n}_{I1}$ and $\mathbf{n}_{I2}$. In particular, for a string which is uniform in its reference configuration we may substitute eqns (3), (9), (14), (15b) and (16) into eqns (4) to obtain

$$\mathbf{n}_{I1} = -\frac{1}{2}\mathbf{f}_I + \mathbf{l}_I - \mathbf{k}_I + \frac{1}{12}m_I[(3 + \alpha)\ddot{\mathbf{r}}_I + (3 - \alpha)\ddot{\mathbf{r}}_{I+1}] \tag{18a}$$

$$\mathbf{n}_{I2} = -\frac{1}{2}\mathbf{f}_I - \mathbf{l}_I + \mathbf{k}_I + \frac{1}{12}m_I[(3 - \alpha)\ddot{\mathbf{r}}_I + (3 + \alpha)\ddot{\mathbf{r}}_{I+1}]. \tag{18b}$$

Then, substituting eqns (18) into eqn (17) we deduce $N - 1$ coupling equations of the form

$$\frac{1}{12}[m_{I-1}(3 - \alpha)\ddot{\mathbf{r}}_{I-1} + (m_{I-1} + m_I)(3 + \alpha)\ddot{\mathbf{r}}_I + m_I(3 - \alpha)\ddot{\mathbf{r}}_{I+1}]$$
$$= \frac{1}{2}(\mathbf{f}_{I-1} + \mathbf{f}_I) + (\mathbf{l}_{I-1} - \mathbf{l}_I) - (\mathbf{k}_{I-1} - \mathbf{k}_I) \qquad I = 2, 3, \ldots, N \tag{19}$$

to restrict the $N + 1$ unknowns $\mathbf{r}_I(t)$. The remaining two equations are specified by boundary conditions.

Three types of boundary value problems can be formulated and these are described in detail in Ref. [2]. Here we merely record the boundary conditions which require specification of:

$$\text{either} \quad \mathbf{r}_1(t) \quad \text{or} \quad \mathbf{n}_{11}(t) \tag{20a}$$

and

$$\text{either} \quad \mathbf{r}_{N+1}(t) \quad \text{or} \quad \mathbf{n}_{N2}(t). \tag{20b}$$

Once the assigned force $\mathbf{f}_I$, assigned director couple $\mathbf{l}_I$, and a constitutive equation for $\mathbf{k}_I$ are specified, eqns (19) and appropriate forms of (20a) and (20b) may be solved for the

$N + 1$ unknowns $\mathbf{r}_I(t)$ subject to the initial conditions

$$\mathbf{r}_I(0) = \mathbf{R}_I + \mathbf{U}_I, \qquad \dot{\mathbf{r}}_I(0) = \mathbf{V}_I \qquad (I = 1, 2, \ldots, N + 1) \tag{21a, b}$$

where $\mathbf{R}_I$ are the reference values of $\mathbf{r}_I$, and $\mathbf{U}_I$ and $\mathbf{V}_I$ are the initial displacement and velocity of the point $\mathbf{r}_I$, respectively. The displacement $\mathbf{u}_I(t)$ for arbitrary time is given by

$$\mathbf{u}_I = \mathbf{r}_I - \mathbf{R}_I. \tag{22}$$

After $\mathbf{r}_I$ are known, the contact forces $\mathbf{n}_{I1}$ and $\mathbf{n}_{I2}$ can be determined by eqns (18).

For the examples discussed in the next sections we consider a homogeneous elastic string which in its reference configuration is straight, of length L, and force free. Also, the string is divided into N equal parts. It follows that we may specify

$$D_I = \frac{L}{N}, \qquad \xi_I = \frac{(I-1)L}{N}, \qquad \mathbf{R}_I = \xi_I \mathbf{e}_1 \qquad (I = 1, 2, \ldots, N + 1) \tag{23a–c}$$

where $\mathbf{e}_1$ is one of the unit base vectors of a fixed Cartesian coordinate system with base vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$. Furthermore, we wish to specify the constitutive equation, eqn (13), of the Cosserat point to be consistent with eqn (A9) of the string. To do this it suffices to consider the simple static problem of uniform tension T^* in the string in the absence of body force. Then, eqns (18)–(20) and (A8c) yield the results

$$\mathbf{n}_{I1} = -\mathbf{k}_I, \qquad \mathbf{n}_{I2} = \mathbf{k}_I, \qquad \mathbf{k}_I = T^*\left(\frac{\mathbf{a}^*}{a^*}\right) \qquad (I = 1, 2, \ldots, N). \tag{24a–c}$$

Now, using eqns (1), (23), (A1) and (A4) we have

$$\mathbf{a}^* = \frac{\mathbf{d}_I}{D_I}, \qquad a^* = \frac{d_I}{D_I} \qquad (I = 1, 2, \ldots, N). \tag{25a, b}$$

Thus, with the help of eqns (7), (24), (25) and (A9) we obtain the constitutive equation

$$k_I = Pg^*\left(\frac{d_I}{D_I}\right). \tag{26}$$

Of course, since the material is elastic this constitutive equation is also valid for dynamic problems.

EXAMPLE: A ROTATING CIRCULAR STRING

In this section we consider the example of a closed circular string which rotates in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane about the $\mathbf{e}_3$ axis with angular velocity $\dot{\theta}(t)$ measured positive in the counter-clockwise direction. The exact solution of this problem was obtained by Rosenau and Rubin[4] and is recorded in Appendix B for convenience. Here, we model the string as a closed chain of N $(N \geqq 3)$ directors $\mathbf{d}_I$ $(I = 1, 2, \ldots, N)$ which are associated with N Cosserat points. Let $\mathbf{e}_r$ and $\mathbf{e}_\theta$ be unit base vectors of a polar coordinate system defined by

$$\mathbf{e}_r(\theta) = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \tag{27a}$$

$$\mathbf{e}_\theta(\theta) = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2. \tag{27b}$$

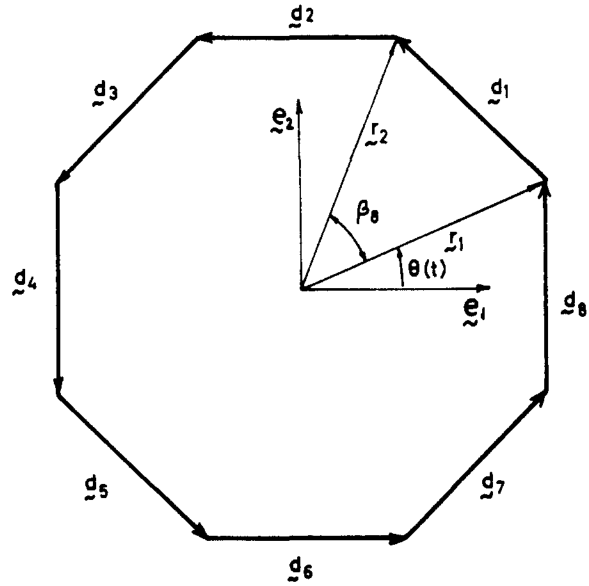

Fig. 2. Model of a circular string using eight Cosserat points showing: the directors $\mathbf{d}_I$ $(I = 1, 2, \ldots, 8)$; the end points $\mathbf{r}_1$ and $\mathbf{r}_2$; the angle β_N $(N = 8)$ between the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$; and the angle $\theta(t)$ between the vector $\mathbf{r}_1$ and the fixed $\mathbf{e}_1$ axis.

Then, for this problem the vectors $\mathbf{r}_I$ and $\mathbf{r}_{I+1}$ may be written in the forms

$$\mathbf{r}_I = r_N \mathbf{e}_r(\phi_I), \qquad \mathbf{r}_{I+1} = r_N \mathbf{e}_r(\phi_I + \beta_N) \tag{28a, b}$$

$$\phi_I(t) = \theta(t) + (I - 1)\beta_N \tag{28c}$$

where $\phi_I(t)$ is the angle between the vector $\mathbf{r}_I$ and the $\mathbf{e}_1$ axis, $r_N(t)$ is the length of each vector $\mathbf{r}_I$, and β_N is the constant angle between the vectors $\mathbf{r}_I$ and $\mathbf{r}_{I+1}$ (Fig. 2). From the geometry of the closed chain of N directors it follows that

$$d_N = 2r_N \sin\frac{\beta_N}{2}, \qquad \beta_N = \frac{2\pi}{N} \qquad (N \geq 3) \tag{29a, b}$$

where we have used eqns (1c) and (3).

Now, in the absence of assigned force ($\mathbf{f}_I = 0$) and assigned director couple ($\mathbf{l}_I = 0$) and with the help of eqns (7), (15), (23), (26), (28) and (29), it may be shown that the $N - 1$ coupling equations (19) yield two scalar equations:

$$\ddot{r}_N - r_N\dot{\theta}^2 + \left(\frac{D_N}{2\sin\dfrac{\beta_N}{2}}\right)\left(\frac{4\pi^2 P}{\rho_0^* L^2}\right) f(\beta_N, \alpha) g^*(a) = 0 \tag{30a}$$

$$2\dot{r}_N\dot{\theta} + r_N\ddot{\theta} = 0 \tag{30b}$$

where the stretch a and the function $f(\beta_N, \alpha)$ are defined by

$$a = \frac{d_N}{D_N}, \qquad f(\beta_N, \alpha) = \frac{6}{[(3 + \alpha) + (3 - \alpha)\cos\beta_N]}\left[\frac{\sin\dfrac{\beta_N}{2}}{\dfrac{\beta_N}{2}}\right]^2. \tag{31a, b}$$

Furthermore, it can be shown that eqns (18) yield

$$\mathbf{n}_{I1} = -Pg^*(a)h(\beta_N, \alpha)\mathbf{e}_\theta(\phi_I) \tag{32a}$$

$$\mathbf{n}_{I2} = Pg^*(a)h(\beta_N, \alpha)\mathbf{e}_\theta(\phi_{I+1}) \tag{32b}$$

where the function $h(\beta_N, \alpha)$ is defined by

$$h(\beta_N, \alpha) = \frac{6\cos\frac{\beta_N}{2}}{[(3+\alpha) + (3-\alpha)\cos\beta_N]}. \tag{33}$$

Using eqns (28), (29b) and (32) it follows that this solution automatically satisfies the two boundary conditions

$$\mathbf{r}_1 = \mathbf{r}_{N+1}, \qquad \mathbf{n}_{11} + \mathbf{n}_{N2} = 0 \tag{34a, b}$$

which require the chain of directors to be closed.

To compare the Cosserat solution with the exact solution of Appendix B we note that, with the help of eqns (29a) and (31a), eqns (30) may be written in terms of the stretch a and eqn (30b) may be integrated to obtain

$$\ddot{a} - \frac{b_3^2}{a^3} + \left(\frac{4\pi^2 P}{\rho_0^* L^2}\right) f(\beta_N, \alpha) g^*(a) = 0 \tag{35a}$$

$$\dot{\theta} = \frac{b_3}{a^2} \tag{35b}$$

where b_3 is a constant of integration. Given values for ρ_0^*, L, P, N ($N \geqq 3$), a functional form for the constitutive equation $g^*(a)$, and given initial conditions for a, $\dot{a}$, θ, $\dot{\theta}$, the constant b_3 in eqn (35b) is determined and the ordinary differential equations (35) may be integrated to obtain $a(t)$, $\theta(t)$. Then the contact forces $\mathbf{n}_{I1}$, $\mathbf{n}_{I2}$ can be obtained form eqns (32).

Physically, eqn (35b) represents conservation of angular momentum about the $\mathbf{e}_3$ axis. Also, the results, eqns (32a) and (32b), indicate that the contact forces $\mathbf{n}_{I1}$ and $\mathbf{n}_{I2}$ always act in the tangential direction (as they should), even though the directors $\mathbf{d}_I$ are tangent to the circular string only in the limit that N approaches infinity. Comparing eqns (35a) and (35b) with eqns (B2a) and (B2b), respectively, we observe that for the same initial conditions and an arbitrary constitutive equation for g^*, eqns (35a) and (35b) will predict exact values for the functions $\dot{a}(t)$ and $\theta(t)$ if the function $f(\beta_N, \alpha)$ equals unity. Furthermore, we observe from eqns (32) that the magnitude of the contact force will also be exact if the function $h(\beta_N, \alpha)$ equals unity. Therefore, the accuracy of the Cosserat solution is controlled by the functions $f(\beta_N, \alpha)$ and $h(\beta_N, \alpha)$. Using eqns (29b), (31b) and (33) we have plotted the values of these functions in Fig. 3 when director inertia is included ($\alpha = 1$) and excluded ($\alpha = 0$). The results in the figure show that the Cosserat solution is quite accurate when the string is modelled by as few as 10 directors and that the effect of director inertia is significant for the rather crude approximations of the string (low values of N).

EXAMPLE: A ROTATING STRAIGHT STRING

In this section we consider the example of a straight string which rotates in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane about the $\mathbf{e}_3$ axis with constant angular velocity ω. The end $\xi_1 = 0$ is fixed at the origin and the end $\xi_{N+1} = L$ is free of contact force. The exact solution of this problem was obtained by Rosenau and Rubin[4] and is recorded in Appendix C for convenience.

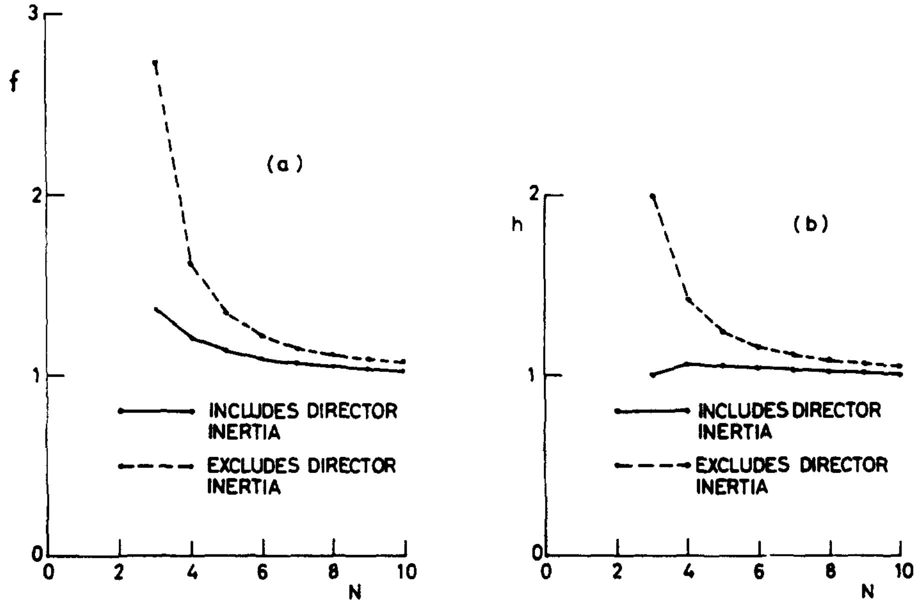

Fig. 3. Rotating circular string: (a) plots of the function $f(\beta_N, \alpha)$ defined in eqn (31b); and (b) plots of the function $h(\beta_N, \alpha)$ defined in eqn (33); when director inertia is included ($\alpha = 1$) and excluded ($\alpha = 0$). N is the number of Cosserat points used to model the string.

Here, we model the string as a chain of N ($N \geqq 1$) directors $\mathbf{d}_I$ ($I = 1, 2, \ldots, N$) which are associated with N Cosserat points. For this problem the vectors $\mathbf{r}_I$ and $\mathbf{r}_{I+1}$ may be written in the forms

$$\mathbf{r}_I = r_I \mathbf{e}_r(\theta), \qquad \mathbf{r}_{I+1} = r_{I+1}\mathbf{e}_r(\theta) \tag{36a, b}$$

$$\theta = \omega t \tag{36c}$$

where $\theta(t)$ is the angle between the vectors $\mathbf{r}_I$ and the $\mathbf{e}_1$ axis, the base vector $\mathbf{e}_r$ is defined by eqn (27a), and the constants r_I are determined by the solution. For later convenience the displacements U_I may be defined by

$$U_I = r_I - \xi_I \tag{37}$$

where ξ_I is the value of r_I in the reference configuration and is determined by eqn (23b).

Now, in the absence of assigned force ($\mathbf{f}_I = 0$) and assigned director couple ($\mathbf{l}_I = 0$) and with the help of eqns (7), (15), (23), (26) and (36), it may be shown that the $N - 1$ coupling equations (19) yield

$$g^*(a_I) - g^*(a_{I-1}) + \frac{\gamma^2}{12LN}[(3-\alpha)r_{I-1} + 2(3+\alpha)r_I + (3-\alpha)r_{I-1}] = 0$$

$$(I = 2, 3, \ldots, N) \tag{38}$$

where the stretch a_I is determined by

$$a_I = \frac{d_I}{D_I} = \frac{(r_{I+1} - r_I)}{L/N} > 0, \qquad r_{I+1} = r_I + \left(\frac{L}{N}\right)a_I \tag{39a, b}$$

and where the constant γ is a non-dimensionalized angular velocity defined by

$$\gamma^2 = \frac{\omega^2 \rho_0^* L^2}{P}. \tag{40}$$

Substituting eqn (39b) and the constitutive equation, eqn (A10), into eqn (38) we obtain the equations

$$a_I = -\left[\frac{(3-\alpha)\gamma^2}{12N^2}\right] + \left[\left\{\frac{(3-\alpha)\gamma^2}{12N^2}\right\}^2 + b_I\right]^{1/2} \tag{41a}$$

$$b_I = a_{I-1}^2 - \left(\frac{\gamma^2}{6LN}\right)[(3-\alpha)r_{I-1} + (9+\alpha)r_I], \qquad I = 2,3,\ldots,N \tag{41b}$$

where we have used the condition that the stretch a_I is positive. The coupling equations (41a) represent $N-1$ equations to restrict the $N+1$ unknowns r_I. The remaining two equations to restrict r_I are boundary conditions of the form

$$\mathbf{r}_1 = 0, \qquad \mathbf{n}_{N2} = 0. \tag{42a, b}$$

With the help of eqns (7), (15), (18), (23), (26), (36), (39) and (A14) the boundary conditions (42a) and (42b) become

$$r_1 = 0, \qquad a_N^2 - 2\frac{(3+\alpha)\gamma^2}{12N^2}a_N - \left(1 + \frac{\gamma^2}{NL}r_N\right) = 0 \tag{43a, b}$$

and the contact forces $\mathbf{n}_{I1}$ and $\mathbf{n}_{I2}$ in eqns (18a) and (18b) may be expressed as

$$\mathbf{n}_{I1} = n_{I1}\mathbf{e}_r(\theta), \qquad \mathbf{n}_{I2} = n_{I2}\mathbf{e}_r(\theta) \tag{44a, b}$$

$$n_{I1} = -\frac{P}{2}\left[a_I^2 + \frac{2(3-\alpha)\gamma^2}{12N^2}a_I - \left(1 - \frac{\gamma^2}{NL}r_I\right)\right] \tag{44c}$$

$$n_{I2} = \frac{P}{2}\left[a_I^2 - \frac{2(3+\alpha)\gamma^2}{12N^2}a_I - \left(1 + \frac{\gamma^2}{NL}r_I\right)\right] \qquad (I = 1,2,\ldots,N). \tag{44d}$$

When the string is modelled by a single Cosserat point ($N = 1$) there are no coupling equations and the solution is determined by eqns (39b), (43a), (43b), (44c) and (44d) so that

$$r_1 = 0, \qquad r_2 = La_1 \tag{45a, b}$$

$$a_1 = \frac{(3+\alpha)\gamma^2}{12} + \left[\left\{\frac{(3+\alpha)\gamma^2}{12}\right\}^2 + 1\right]^{1/2} \tag{45c}$$

$$n_{11} = -\frac{P}{2}\gamma^2 a_1, \qquad n_{12} = 0. \tag{45d, e}$$

To determine the solution when the string is modelled by more than one Cosserat point ($N > 1$) we: (a) guess a value for a_1; (b) use eqns (39b), (41a), (41b) and (43a) to determine

Table 1. Rotating straight string

	Exact solution		($N = 1$) Cosserat solution including director inertia		($N = 1$) Cosserat solution excluding director inertia	
γ	$\frac{u^*(L)}{L}$	$\frac{T^*(0)}{P}$	$\frac{u_2}{u^*(L)}$	$\frac{-n_{11}}{T^*(0)}$	$\frac{u_2}{u^*(L)}$	$\frac{-n_{11}}{T^*(0)}$
0.1	0.0033400	0.0050209	0.9997	0.9992	0.7494	0.9983
1.0	0.39773	0.74528	0.9741	0.9308	0.7060	0.8593
10.0	68.317	4018.8	0.9614	0.8296	0.7175	0.6223

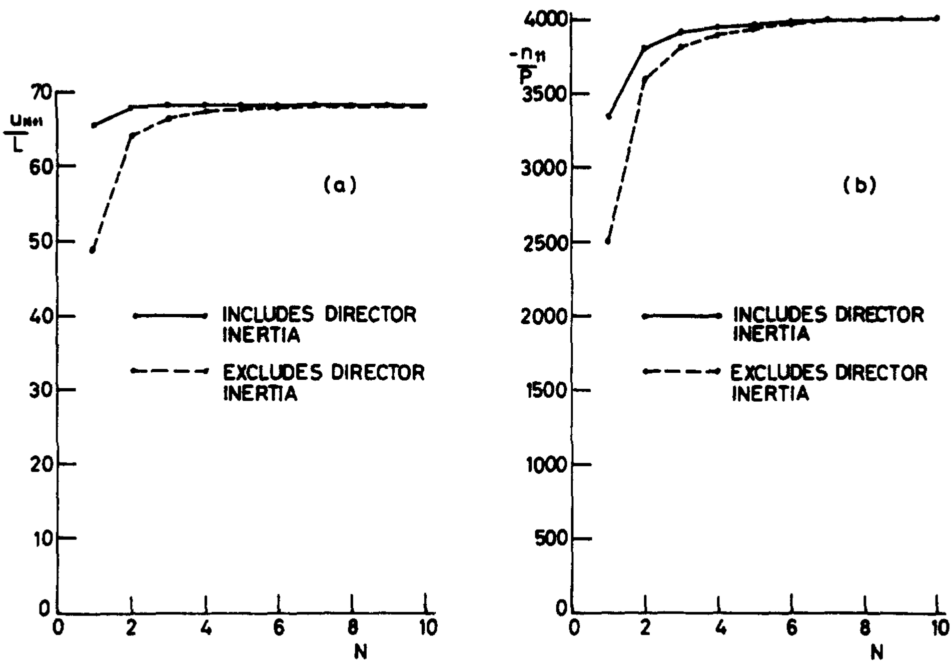

Fig. 4. Large deformation ($\gamma = 10.0$) of a rotating straight string: (a) plots of the displacement u_{N+1} at the free end ($\xi_{N+1} = L$); and (b) plots of the contact force $\mathbf{n}_{11}$ at the fixed end ($\xi = 0$); when director inertia is included ($\alpha = 1$) and excluded ($\alpha = 0$). N is the number of Cosserat points used to model the string.

values for r_I and a_I ($I = 2, 3, \ldots, N$); and (c) iterate on the guess for a_1 until eqn (43b) is satisfied. After values for r_I and a_I are determined we calculate the contact forces $\mathbf{n}_{I1}$ and $\mathbf{n}_{I2}$ from eqns (44).

The solution was obtained for three values of the normalized angular velocity γ corresponding to small deformation ($\gamma = 0.1$), moderate deformation ($\gamma = 1.0$) and large deformation ($\gamma = 10.0$). For each case we modelled the string with 1–10 Cosserat points. The largest errors in the prediction of the displacement and the tension were obtained at the free ($\xi_{N+1} = L$) and the fixed end ($\xi_1 = 0$), respectively. Therefore, in Table 1 we have presented values for these quantities predicted by the exact solution of Appendix C and the solution using a single Cosserat point. From Table 1 we observe that the simple Cosserat solution which includes director inertia is very accurate even for moderate deformation, and that the influence of director inertia is significant. The fast rate of convergence of the Cosserat solution for large deformation ($\gamma = 10.0$) is shown in Fig. 4. Also, the spatial variation of the displacement and tension (n is the magnitude of the contact force) for large deformation ($\gamma = 10.0$) is shown in Fig. 5 for three values of N ($N = 1, 2, 10$). Finally, we note that the curves in Fig. 5 for N = 10 are nearly indistinguishable from those predicted by the exact solution of Appendix C.

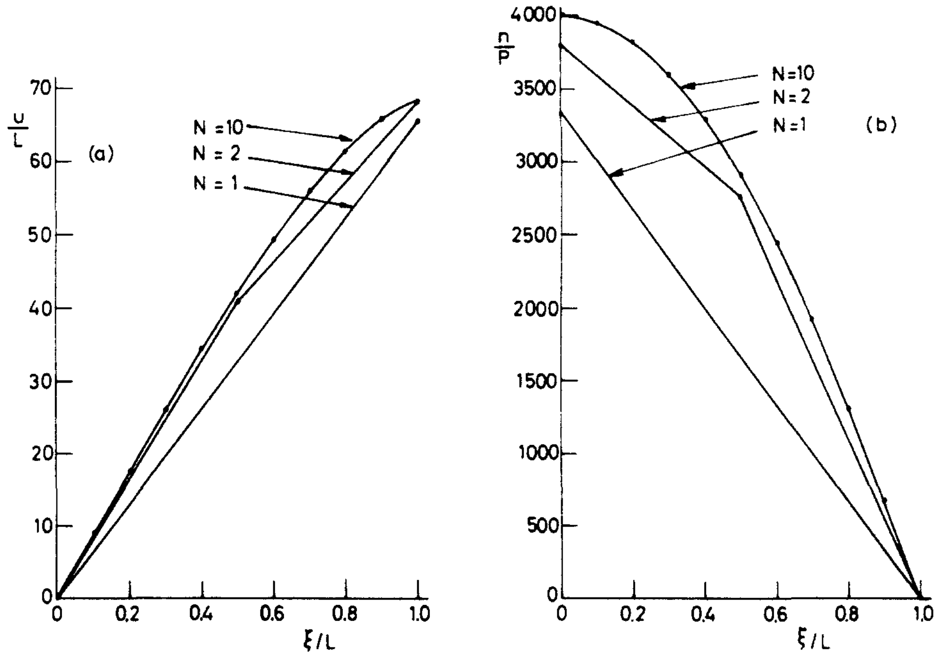

Fig. 5. Large deformation ($\gamma = 10.0$) of a rotating straight string: (a) plots of the displacement u; and (b) plots of the tension (n is the magnitude of the contact force). N is the number of Cosserat points used to model the string and director inertia is included ($\alpha = 1$).

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APPENDIX A: DERIVATION FROM THE THEORY OF A COSSERAT CURVE

In this appendix we briefly review an appropriate form of the theory of a Cosserat curve which models a string. After introducing a kinematic approximation of the position vector as well as a number of definitions we derive the equations of motion of a theory of a Cosserat point appropriate to model a string.

For our purposes we note that a string can be modelled using a theory of a Cosserat curve which is obtained by suppressing the effect of the directors in a Cosserat description of a rod[5,6]. With reference to the present configuration, let material points of the string be identified by the convected coordinate ξ and located by the position vector $\mathbf{r}^*(\xi, t)$. Then a motion of the string is defined by†

$$\mathbf{r}^* = \mathbf{r}^*(\xi, t), \qquad \mathbf{a}^* = \frac{\partial \mathbf{r}^*}{\partial \xi}, \qquad a^* = (\mathbf{a}^* \cdot \mathbf{a}^*)^{1/2} > 0 \tag{A1a–c}$$

where $\mathbf{a}^*$ is a tangent vector to the string curve and a^* is the stretch associated with the space curve. The velocity $\mathbf{v}^*$ of the material point is given by

$$\mathbf{v}^* = \frac{\partial}{\partial t}\mathbf{r}^*(\xi, t) = \dot{\mathbf{r}}^*. \tag{A2}$$

†Throughout this paper we use a superposed (*) to indicate that the quantity is associated with the description of the Cosserat curve.

where a superposed dot denotes material time differentiation holding ξ fixed.

Now the equations of conservation of mass, balance of linear momentum, balance of angular momentum, and a definition of the mechanical power P^* may be written in the forms

$$\lambda^* = \lambda^*(\xi) = \rho^* a^* = \rho_0^* A^* \tag{A3a}$$

$$\lambda^* \dot{\mathbf{v}}^* = \lambda^* \mathbf{f}^* + \frac{\partial \mathbf{n}^*}{\partial \xi} \tag{A3b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\lambda^*(\mathbf{r}^* \times \mathbf{v}^*)] = \mathbf{r}^* \times \lambda^* \mathbf{f}^* + \frac{\mathrm{d}}{\mathrm{d}\xi}[\mathbf{r}^* \times \mathbf{n}^*] \tag{A3c}$$

$$P^* = \lambda^* \mathbf{f}^* \cdot \mathbf{v}^* + \frac{\mathrm{d}}{\mathrm{d}\xi}[\mathbf{n}^* \cdot \mathbf{v}^*] - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\lambda^* \mathbf{v}^* \cdot \mathbf{v}^*\right] \tag{A3d}$$

respectively, where $\rho^*(\xi, t)$ is the mass density (mass per unit length) in the present configuration; ρ_0^* and A^* are the reference values of ρ^* and a^*, respectively; $\mathbf{n}^*(\xi, t)$ is the contact force; and $\mathbf{f}^*$ is the assigned external force.

To derive equations of the forms of eqns (4), (5) and (11) associated with the theory of a Cosserat point, we assume that the position vector $\mathbf{r}^*$ depends at most linearly on the coordinate ξ. Hence, it admits the representation

$$\mathbf{r}^*(\xi, t) = \bar{\mathbf{r}}_I(t) + \theta_I \mathbf{d}_I(t) \tag{A4a}$$

$$\theta_I = \frac{(2\xi - \xi_I - \xi_{I+1})}{2(\xi_{I+1} - \xi_I)}, \qquad \xi \in [\xi_I, \xi_{I+1}] \tag{A4b, c}$$

where $\bar{\mathbf{r}}_I$ and $\mathbf{d}_I$ are the same quantities introduced in the main text of the paper, and where ξ_I is defined so that $\theta_I = \pm 1/2$ on the ends $\xi = \xi_{I+1}$ and ξ_I, respectively. In view of eqns (A2) and (A4a) the velocity $\mathbf{v}^*$ may be represented in the form

$$\mathbf{v}^* = \bar{\mathbf{v}}_I + \theta_I \mathbf{w}_I \tag{A5}$$

where $\bar{\mathbf{v}}_I$ and $\mathbf{w}_I$ are defined in eqns (2a) and (2b), respectively. Next, let us relate the quantities m_I, y_I^{11}, y_I^{11}, $\mathbf{f}_I$, $\mathbf{n}_{I1}, \mathbf{n}_{I2}$, $\mathbf{m}_{I1}$, $\mathbf{m}_{I2}$, $\mathbf{l}_I$, $\mathbf{k}_I$ and P_I to the quantities defined above through the equations:

$$m_I = \int_{\xi_I}^{\xi_{I+1}} \lambda^* \,\mathrm{d}\xi, \qquad m_I y_I^1 = \int_{\xi_I}^{\xi_{I+1}} \lambda^* \theta_I \,\mathrm{d}\xi \tag{A6a, b}$$

$$m_I y_I^{11} = \int_{\xi_I}^{\xi_{I+1}} \lambda^* \theta_I^2 \,\mathrm{d}\xi \tag{A6c}$$

$$\mathbf{n}_{I1} = -[\mathbf{n}^*]_{\xi=\xi_I} = -\mathbf{n}^*(\xi_I, t) \tag{A6d}$$

$$\mathbf{n}_{I2} = [\mathbf{n}^*]_{\xi=\xi_{I+1}} = \mathbf{n}^*(\xi_{I+1}, t) \tag{A6e}$$

$$\mathbf{m}_{I1} = -[\theta_I \mathbf{n}^*]_{\xi=\xi_I} = -\frac{1}{2}\mathbf{n}_{I1} \tag{A6f}$$

$$\mathbf{m}_{I2} = [\theta_I \mathbf{n}^*]_{\xi=\xi_{I+1}} = \frac{1}{2}\mathbf{n}_{I2} \tag{A6g}$$

$$\mathbf{f}_I = \int_{\xi_I}^{\xi_{I+1}} \lambda^* \mathbf{f}^* \,\mathrm{d}\xi, \qquad \mathbf{l}_I = \int_{\xi_I}^{\xi_{I+1}} \lambda^* \mathbf{f}^* \theta_I \,\mathrm{d}\xi \tag{A6h, i}$$

$$\mathbf{k}_I = \left(\frac{1}{\xi_{I+1} - \xi_I}\right) \int_{\xi_I}^{\xi_{I+1}} \mathbf{n}^* \,\mathrm{d}\xi, \qquad P_I = \int_{\xi_I}^{\xi_{I+1}} P^* \,\mathrm{d}\xi. \tag{A6j, k}$$

Now substituting the representations (A4a) and (A5) into eqns (A3) and using definitions (A6) we can derive the equations of conservation of mass (4a), balance of linear momentum (4b), balance of angular momentum (5), and the mechanical power expression (10), by integrating over the material region $\xi \in [\xi_I, \xi_{I+1}]$. The director momentum eqn (4c) can be obtained by multiplying eqn (A3b) by θ_I and integrating over the material region $\xi \in [\xi_I, \xi_{I+1}]$ to obtain

$$\frac{d}{dt}\int_{\xi_I}^{\xi_{I+1}} \lambda^* \mathbf{v}^* \theta_I \, d\xi = \int_{\xi_I}^{\xi_{I+1}} \lambda^* \mathbf{f}^* \theta_I \, d\xi + [\theta_I \mathbf{n}^*]_{\xi=\xi_{I+1}} - [\theta_I \mathbf{n}^*]_{\xi=\xi_I} - \left(\frac{1}{\xi_{I+1}-\xi_I}\right)\int_{\xi_I}^{\xi_{I+1}} \mathbf{n}^* \, d\xi. \tag{A7}$$

Then, substituting eqn (A5) into eqn (A7) and using definitions (A6) we derive eqn (4c).

For an elastic material we assume the existence of a strain energy function ψ^* and it can be shown that

$$\psi^* = \psi^*(a^*, A^*), \qquad P^* = \mathbf{n}^* \cdot \dot{\mathbf{a}}^* = \lambda^* \dot{\psi}^* \tag{A8a, b}$$

$$\mathbf{n}^* = T^*\left(\frac{\mathbf{a}^*}{a^*}\right), \qquad T^* = \lambda^* \frac{\partial \psi^*}{\partial a^*} \tag{A8c, d}$$

where we note that ψ^* also depends on ξ when the string is inhomogeneous. Using these results it is relatively simple to show that eqn (A8c) ensures that the angular momentum eqn (A3c) is automatically satisfied. Also, by substituting eqn (A8c) into eqn (A3b) and taking $A^* = 1$ we obtain the usual equations of a string.

For our purposes we consider a homogeneous string which is characterized by the constitutive equation

$$T^* = P g^*(a^*) \tag{A9}$$

where P is a constant having the units of force and g^* is a function to be specified. A particular form for g^* appropriate for a non-linear string which is force free in its reference configuration may be specified by

$$g^* = \frac{1}{2}[(a^*)^2 - 1]. \tag{A10}$$

This form was used for the examples considered by Rosenau and Rubin[4].

APPENDIX B: A ROTATING CIRCULAR STRING

In this appendix we consider the motion of a circular string rotating in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane about the $\mathbf{e}_3$ axis with angular velocity $\dot{\theta}^*(t)$. The position vector $\mathbf{r}^*(\xi, t)$ of material points on this string may be written in the form

$$\mathbf{r}^* = r^*(t)\mathbf{e}_r\left(\frac{2\pi\xi}{L} + \theta^*(t)\right) \tag{B1a}$$

$$a^*(t) = \frac{2\pi}{L} r^*(t) \tag{B1b}$$

where r^* and a^* are, respectively, the radius and stretch of the string in the present configuration, and $\mathbf{e}_r$ is the base vector defined by eqn (27a).

Using eqns (13), (14), (15) and (18) of Rosenau and Rubin[4] and eqn (A9) of this paper, the motion of the string is characterized by

$$\ddot{a}^* - \frac{b_3^2}{(a^*)^3} + \left(\frac{4\pi^2 P}{\rho_0^* L^2}\right) g^*(a^*) = 0 \tag{B2a}$$

$$\dot{\theta}^* = \frac{b_3}{(a^*)^2} \tag{B2b}$$

where b_3 is a constant. In deriving eqn (B2a) we have retained the general form of g^* given in eqn (A9). Also, we note that the quantities a^*, θ^*, g^* correspond to the quantities a_1, θ_1, g_1 defined by Rosenau and Rubin[4].

Given a functional form for g^* and initial conditions for a^*, $\dot{a}^*$, θ^*, $\dot{\theta}^*$ the constant b_3 is determined and the ordinary differential equations (B2) may be integrated to obtain $a^*(t)$ and $\theta^*(t)$ which determine the motion of the string.

APPENDIX C: A ROTATING STRAIGHT STRING

In this appendix we consider the motion of a straight string which is rotating in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane with constant angular velocity ω about the $\mathbf{e}_3$ axis. The end $\xi = 0$ is fixed at the origin and the end $\xi = L$ is free of contact force. The position vector $\mathbf{r}^*(\xi, t)$ of material points on this string may be written in the form

$$\mathbf{r}^* = r^*(\xi)\mathbf{e}_r(\theta^*(t)) \tag{C1a}$$

$$a^*(\xi) = \frac{\partial r^*}{\partial \xi}, \qquad \theta^*(t) = \omega t \tag{C1b, c}$$

where r^* and a^* are, respectively, the radial distance of the material point ξ from the fixed end and the stretch of the string. Recalling eqns (13c), (30) and (32) of Rosenau and Rubin[4], the function g^* is determined by the integral

$$\int_0^{g^*} \frac{\mathrm{d}\lambda}{\left[b_4 - \frac{2}{3}\left(\frac{\gamma}{L}\right)^2 (1 + 2\lambda)^{3/2}\right]^{1/2}} = L - \xi \tag{C2a}$$

$$\gamma^2 = \frac{\rho_0^* \omega^2 L^2}{P} \tag{C2b}$$

where the constant b_4 is given by

$$b_4 = \frac{2}{3}\left(\frac{\gamma}{L}\right)^2 [1 + 2g^*(a^*(0))]^{3/2} = \frac{2}{3}\left(\frac{\gamma}{L}\right)^2 [a^*(0)]^3 \tag{C3}$$

and where use has been made of the boundary condition that the string is free of contact force ($g^* = 0$) at $\xi = L$. In deriving eqns (C2) and (C3) the specific form, eqn (A10), was used for the function g^*. Also, we note that the quantities a^*, θ^*, g^* correspond to the quantities a_2, θ_1, g_2 of Rosenau and Rubin[4].

For our purposes, it is more convenient to write eqn (C2) in terms of $a^*(\xi)$. This can be done by letting $\beta = (1 + 2\lambda)^{1/2}$ and using eqns (A10) and (C3) to obtain

$$\int_1^{a^*(\xi)} \frac{\beta\,\mathrm{d}\beta}{[(a^*(0))^3 - \beta^3]^{1/2}} = \left(\frac{2}{3}\right)^{1/2} \gamma\left(1 - \frac{\xi}{L}\right). \tag{C4}$$

For a given value of γ the solution is obtained by evaluating eqn (C4) at $\xi = 0$ and solving the resulting equation for $a^*(0)$. Then $a^*(\xi)$ can be obtained from eqn (C4) by integration. To obtain $r^*(\xi)$ we integrate eqn (C1b) subject to the boundary condition that the end $\xi = 0$ is fixed at the origin ($r^*(0) = 0$) to obtain

$$r^*(\xi) = \int_0^{\xi} a^*(\lambda)\,\mathrm{d}\lambda. \tag{C5}$$

The solution of eqns (C4) and (C5) was evaluated numerically. To avoid the singularity in eqn (C4) at $\beta = a^*(0)$ we introduced the change of variables

$$\beta^3 = [a^*(0)]^3 \sin^2 \eta, \qquad \eta(\xi) = \sin^{-1}\left[\left(\frac{a^*(\xi)}{a^*(0)}\right)^{3/2}\right] \tag{C6a, b}$$

to rewrite eqn (C4) in the form

$$\int_{\eta(L)}^{\eta(\xi)} (\sin \eta)^{1/3}\,\mathrm{d}\eta = \left[\frac{3}{2a^*(0)}\right]^{1/2} \gamma\left(1 - \frac{\xi}{L}\right). \tag{C7}$$

Then a solution was obtained by: (a) guessing a value of $a^*(0)$; (b) evaluating the integral in eqn (C7) using the trapezoidal rule; and (c) iterating on the initial guess of $a^*(0)$ until eqn (C7) is satisfied for $\xi = 0$. Finally, g^* and r^* were obtained from eqns (A10) and (C5), respectively.